WHEN IS THE FUNCTIONS MEASURABLE?

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Abstract

In this article we explore under which conditions on the interior function the composition of functions is measurable. We also study the sharpness of the result by providing a counterexample for weaker hypotheses.

Keywords: measurable functions, composition, Lebesgue measure

1 Introduction

It is a well known fact among young analysts that the *composition of measurable functions is not necessarily measurable,* although this fact, to be true, has to be stated precisely. It is therefore necessary to first start with the definition of measurable function.

Definition 1.1 Let $(X,M),(X,N)$ be measurable spaces. We say $f:(X,M)\to(X,N)$ is measurable if $f^{-1}(A)\in M$ for every $A\in N$.

It is clear then, from this definition, that whether a function is measurable or not depends on the measurable spaces chosen. Let a and b be the Lebesgue and Borel σ- algebras on $\mathbb{R}$ respectively. In general, if $f,g:(\mathbb{R},a)\to(\mathbb{R},b)$ are measurable $g\circ f:(\mathbb{R},a)\to(\mathbb{R},b)$ does not have to be so. In order for $g\circ f:(\mathbb{R},a)\to(\mathbb{R},b)$ to be measurable (this last condition hold when g is continuous), but in general this is not the case.

Here we are interested in sufficient conditions on $f:(\mathbb{R},a)\to(\mathbb{R},b)$ that can guarantee that if $g:(\mathbb{R},a)\to(\mathbb{R},b)$ is measurable, so is $g\circ f:(\mathbb{R},a)\to(\mathbb{R},a)$. We will consider then m^* and m, the Lebesgue exterior measure and measure respectively. For brevity, we will speak of a/b and a/a measurable functions.

Rather counterintuitively, great regularity or monotonicity of f does not guarantee that the composition $g\circ f$ will be a/b measurable when g is. We illustrate this point with the following example.

Example 1.2. We will first proceed to construct a strictly increasing C^{∞} function f which is not a/a measurable, that is, such that there exists $D\in a$ such that $f^{-1}(D)\notin a$.

Let $C \subset I := [0,1]$ be a Smith-Volterra-Cantor set such that $m(C) \in (0,1)$ [3, p.39] and consider the function

$$y(x) := e^{-(1-x^2)^{-1}}$$

for $x \in (-1,1)$. $y \in \mathcal{C}((-1,1),\mathbb{R})$. Since C is closed, $I \setminus C$ is open, so it has a sountable number of connected components each of which is an open interval. Let us define $h(x) = 0$ if $x \in C$ and

$$h(x) = 2^{-(b-a)^{-1}} y\left(\frac{2x-b-a}{b-a}\right)$$

if $x \in (a,b)$ where (a,b) is a connected component of $I \setminus C$. Let $M_n := \max\left|y^{(n)}\right|$. We will show now that $h \in \mathcal{C}^{\infty}(I,\mathbb{R})$. It is clear that h is $\mathcal{C}^{\infty}$ in $I \setminus C$. If $x \in C$, let us check that $\lim_{y \to x^-} f(x) = 0$ (in case the limit can be taken). If $x = b$ for some $(a,b) \subset I \setminus C$, this is obvious. Otherwise, there is a sequence of points in C converging to x from the left. Thus, given $\varepsilon \in \mathbb{R}^+$, there exists $y \in C$, $x - \varepsilon < y < x$, so any $z \in (y,x) \setminus C$ belongs to an interval (a,b) with $b - a < \varepsilon$ and, therefore,

$$h(z) = 2^{-(b-a)^{-1}} y\left((2x-b-a)/(b-a)\right) < 2^{-(b-a)^{-1}} < b-a < \varepsilon.$$

Hence, $\lim_{y \to x^-} f(x) = 0$. Repeating the argument for limits from the right and observing that $h \setminus c = 0$, we conclude that h is continuous. Assuming h is $n-1$ times differentiable and taking into account that

$$\left|h^{(n)}(x)\right| = 2^{-(b-a)^{-1}} 2^n (b-a)^{-n} \left|y^{(n)}\left(\frac{2x-b-a}{b-a}\right)\right| \le 2^{-(b-a)^{-1}} 2^n (b-a)^{-n} M_n,$$

for any $x \in (a,b) \subset I \setminus C$, we can reason as before to conclude that $h \in \mathcal{C}^{\infty}(I,\mathbb{R})$.

Define $f(x) = \int_0^y h(x)dy$. $f \in \mathcal{C}^{\infty}(I,\mathbb{R})$ and f is strictly increasing. Indeed, since C is totally disconnected, given $x,y \in I$, $x < y$, there exist $t,s \in (x,y)$, $t < s$ such that $[t,s] \subset I \setminus C$, so $h(x) > 0$ in $[t,s]$ and hence $f(y) - f(x) = \int_x^y h(z)dz > 0$.

Given a connected component (a,b) of $I \setminus C$,

$$m\left(f\left(a,b\right)\right)=f\left(b\right)-f\left(a\right)=\int_a^b h\left(z\right)dz.$$

Thus, given that f is strictly increasing, $I \setminus C$ has a countable number of connected components and that the Lebesgue measure is σ - additive, $m\left(f\left(I \setminus C\right)\right)=\int_{I \setminus C} h\left(z\right)dz$. $f\left(I\right)$ is also measurable because it is an interval. Since f is strictly increasing, $f\left(C\right)\cap f\left(I \setminus C\right)=\emptyset$ and $f\left(C\right)=f\left(I\right)\setminus f\left(I \setminus C\right)$ and, therefore, $f\left(C\right)$ is measurable. Hence,

$$m\left(f\left(C\right)\right)=m\left(f\left(I \setminus C\right)\right)=f\left(1\right)-f\left(0\right)-\int_{I \setminus C} h\left(z\right)dz=\int_0^1 h\left(z\right)dz-\int_{I \setminus C} h\left(z\right)dz=\int_C h\left(z\right)dz$$

Now, since $m\left(C\right)>0$, there exists $D \subset C$ such that $D \notin \alpha$ [3, Exercise 29, pg. 39]. We have that $f\left(D\right)\subset f\left(C\right)$, so $m^*\left(f\left(D\right)\right)\leq m^*\left(f\left(C\right)\right)=0$ and, therefore, $m^*\left(f\left(D\right)\right)=0$, so $f\left(D\right)\in \alpha$. Finally, $f^{-1}\left(f\left(D\right)\right)=D \notin \alpha$, so f is not α / α measurable.

We now define a α / β measurable function g in such a way that $f \circ g$ is not α / β measurable. Let g be the characteristic function of the set $f\left(D\right)$. g is measurable since $f\left(D\right)$ is. Furthermore, $\left\{1\right\}\in \beta$, but $\left(f \circ g\right)^{-1}\left(\left\{1\right\}\right)=f^{-1}\left(f\left(D\right)\right)=D \notin \alpha$, so $g \circ f$ is not α / β measurable.

The Example 1.2 has shown that, if we are to provide sufficient conditions for $g \circ f$ to be α / α measurable we will have to look further away than the regularity of f. In fact, if is clear from the example that the behavior of f, when it takes inverse images is determinant on the behavior of the composition, so we will try first to impose conditions on f^{-1} in the case f is invertible.

For the next result we consider $\Omega, \Lambda \in \alpha$, $f:\left(\Omega, \alpha\right)\to\left(\mathbb{R}, \beta\right)$, $g:\left(\Lambda, \alpha\right)\to\left(\mathbb{R}, \beta\right)$, $f\left(\Omega\right)\in \Lambda$.

Lemma 1.3. *If g is α / β measurable, f invertible and f^{-1} is absolutely continuous, then $g \circ f$ is α / β measurable.*

Proof. Let $B \in \beta$. Then $g^{-1}(B) \in \alpha$. Since f^{-1} is absolutely continuous it takes α sets to α sets [7 p. 250], so $f^{-1}\left(g^{-1}(B)\right) \in \alpha$.

Observe that Lemma 1.3 crucially avoids the circumstances of Example 1.2, since, in that case, the function f^{-1} was not absolutely continuous (since it did not map α sets to α sets). This illustrates that, in general, even if f is absolutely continuous, f^{-1} needs not to be –see [1,9]. A necessary sufficient sondition for f^{-1} to be absolutely continuous (in the case the domain is an interval) can be found in the following result –cf. [2, Lemma 2.2], [9].

$$\sum_{n=m+1}^{\infty} (b_n - a_n) < \delta.$$

Let $X = \bigcup_{n=1}^{m} [a_n, b_n)$. Then, $\mu^*(E_k \setminus X) < \delta$ since $\{[a_n, b_n)\}_{n=m+1}^{\infty}$ is a cover of $E_k \setminus X$. Thus,

Take $x \in U \setminus V$. Then there exist $r \in \mathbb{R}^+$ such that $(x - r, x + r) \subset U$. Since $x \in C, x \in \partial(I \setminus C)$, so there exists $y \in (x - r, x + r) \setminus C$. Since $I \setminus C$ is open, there exists $s \in \mathbb{R}^+$ such that $(y - s, y + s) \subset (x - r, x + r) \setminus C$. Thus, $\mu(\cap(x - r, x + r)) = \mu(((x - r, x + r) \setminus V) \cup W) = 2r$. Therefore, $\mu((x - r, x + r) \setminus C) = 0$ and, since $(y - s, y + s) \subset (x - r, x + r) \setminus C$, we have that $\mu((y - s, y + s)) = 0$, which is a contradiction.

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